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On the connection between sets of operator synthesis and sets of spectral synthesis for locally compact groups

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Abstract

We extend the results by Froelich and Spronk and Turowska on the connection between operator synthesis and spectral synthesis for $A(G)$ to second countable locally compact groups G. This gives us another proof that one-point subset of G is a set of spectral synthesis and that any closed subgroup is a set of local spectral synthesis. Furthermore, we show that "non-triangular" sets are strong operator Ditkin sets and we establish a connection between operator Ditkin sets and Ditkin sets. These results are applied to prove that any closed subgroup of G is a local Ditkin set.

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1. Introduction

In [A], Arveson discovered a connection between the invariant subspace theory and spectral synthesis. He defined (operator) synthesis for subspace lattices and proved

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the failure of operator synthesis by using the famous example of Schwartz on non-synthesizability of the two-sphere S^2 for $A(\mathbb{R}^3)$. In [F], Froelich made this connection more precise for separable abelian group. For G a separable compact group this relation was obtained in [ST, Theorem 4.6]. We generalize these results to second countable locally compact groups (Theorems 4.3 and 4.11). We use the definition of sets of operator synthesis as defined in [ShT1]. We prove that a closed subset $E \subset G$ is set of local spectral synthesis for $A(G)$ if and only if the diagonal set $E^* = \{(s,t) \in G \times G \mid st^{-1} \in E\}$ is a set of operator synthesis with respect to Haar measure. We remark that if $A(G)$ has an (unbounded) approximate identity then any set of local spectral synthesis is a set of synthesis for $A(G)$ and any compact set of local spectral synthesis is globally spectral for any group G. We give a new proof that a one-point set is spectral and any closed subgroup of second countable group is a set of local spectral synthesis. Using operator synthesizability of sets of finite width we obtain certain example of sets of spectral synthesis.

Operator-Ditkin sets have been defined in [ShT1]. In Section 5, we show first that "non-triangular" type sets are strong operator-Ditkin. The proof is inspired by Drury [D]. For G second countable group we prove that the diagonal set $E^* \subset G \times G$ is a strong operator-Ditkin set with respect to Haar measure then $E \subset G$ is a local Ditkin set for $A(G)$ and conversely, for any strong Ditkin set $E \subset G$, the set E^* is operator-Ditkin with respect to Haar measure. As an application we obtain that any closed subgroup of a second countable group is a local Ditkin set. This result was known for neutral subgroups of arbitrary locally compact groups G [DD] and amenable groups G [FKLS].

2. Preliminaries and notations

Let G be a locally compact σ-compact separable group with left Haar measure $m_G = dg$. Let $L^p(G)$, $p = 1, 2$, denote the space of p-integrable functions with norm $\|\cdot\|_p$ and let $C_c(G)$ denote the algebra of continuous compactly supported complex-valued functions on G. The convolution algebra $L^1(G)$ is an involutive algebra with involution defined by $f^*(s) = \Delta^{-1}(s)f(s^{-1})$, where Δ is the modulus of the group. Let Σ be the set of all (equivalence classes of) continuous unitary representations π of G in Hilbert spaces H_π. For $f \in L^1(G)$, $\pi \in \Sigma$, we put $\pi(f) = \int_G f(g)\pi(g)\,dg \in B(H_\pi)$ with the integral converging in the strong operator topology, and then

$$\|f\|_\Sigma = \sup_{\pi\in\Sigma} \|\pi(f)\|,$$

where $\|\cdot\|$ is the operator norm in $B(H_\pi)$.

The *enveloping* C^**-algebra* $C^*(G)$ of G is the completion of $L^1(G)$ with respect to $\|\cdot\|_\Sigma$. Let $\lambda : G \to B(L^2(G))$ be the left regular representation given by $\lambda(s)f(g) = f(s^{-1}g)$. We denote by $C_r^*(G)$ the *reduced* C^**-algebra* of G, that is the C^*-algebra generated by operators $\lambda(f) \in B(L^2(G))$, $f \in L^1(G)$, and by $VN(G)$ the von Neumann algebra of G, that is

$$VN(G) = \overline{\text{span}}^{\text{WOT}}\{\lambda(g) : g \in G\} = \overline{C_r^*(G)}^{\text{WOT}} \subset B(L^2(G)).$$

The *Fourier–Stieltjes algebra*, $B(G)$, is the set of all coefficients $s \mapsto \pi_{\xi,\eta}(s) = (\pi(s)\xi, \eta)$, where $\pi \in \Sigma$, ξ, $\eta \in H_\pi$, of unitary representations of G, as defined by Eymard [E]. $B(G)$ is a Banach algebra with respect to the norm

$$\|u\| = \inf\{\|\xi\| \ \|\eta\| : u = \pi_{\xi,\eta}\}.$$

Note that $B(G) \simeq C^*(G)^*$.

The *Fourier algebra*, $A(G)$, is the family of functions $s \mapsto (\lambda(s)\xi, \eta) = \bar{\eta} * \check{\xi}$, ξ, $\eta \in L^2(G)$, $\check{\xi}(s) = \xi(s^{-1})$. $A(G)$ is identified with the predual $VN(G)_*$ via $\langle(\lambda(s)\xi, \eta), T\rangle = (T\xi, \eta)$, and thus is a normed algebra with the norm denoted by $\|\cdot\|_A$. It is known that for $u \in A(G)$ there exist even $\xi, \eta \in L^2(G)$, such that $u = \bar{\eta} * \check{\xi}$ and $\|u\|_A = \|\xi\|_2 \cdot \|\eta\|_2$. Furthermore, $A(G)$ is a closed ideal in $B(G)$.

Since $A(G)$ is the predual of the von Neumann algebra $VN(G)$, $A(G)$ possesses a structure of operator space and one can define a notion of completely bounded multipliers, $M_{\text{cb}}A(G)$, for $A(G)$. For the theory of operator space and completely bounded maps we refer the reader to [EfR,BSm,S]. A complex-valued function $u : G \to \mathbb{C}$ is a completely bounded multiplier if it is a multiplier, i.e. $uA(G) \subset A(G)$, and is completely bounded as a linear map on $A(G)$. We have $A(G) \subset B(G) \subset M_{\text{cb}}A(G)$ and, for $u \in A(G)$, $\|u\|_{\text{cb}} \leqslant \|u\|_A$, where $\|\cdot\|_{\text{cb}}$ is the completely bounded norm (see [S, Corollary 2.3.3]). We will use here a characterization of $M_{\text{cb}}A(G)$ obtained by N. Spronk in [S] as formulated in Theorem 3.1.

3. Spectral and operator synthesis

Let A be a semisimple, regular, commutative Banach algebra with X_A as spectrum; for any $a \in A$ we shall denote then by $\hat{a} \in C_0(X_A)$ its Gelfand transform. Let also $E \subset X_A$ be a closed subset. We then denote by

$$I_A(E) = \{a \in A \mid \hat{a}^{-1}(0) \text{ contains } E\},$$

$$J_A^0(E) = \{a \in A \mid \hat{a}^{-1}(0) \text{ contains a neighborhood of } E\} \text{ and } J_A(E) = \overline{J_A^0(E)}.$$

It is known that $I_A(E)$ and $J_A(E)$ are the largest and the smallest closed ideals with E as hull, i.e. if I is a closed ideal such that $\{x \in X_A : f(x) = 0 \text{ for all } f \in I\} = E$ then

$$J_A(E) \subset I \subset I_A(E).$$

Let $I_A^c(E)$ denote the set of all compactly supported functions $f \in I_A(E)$. We say that E is a *set of spectral synthesis* (*local spectral synthesis*) for A if $J_A(E) = I_A(E)$ ($I_A^c(E) \subset J_A(E)$).

Let A^* be the dual of A. For $a \in A$ we set $\operatorname{supp}(a) = \overline{\{x \in X_A : \hat{a}(x) \neq 0\}}$ and $\operatorname{null}(a) = \{x \in X_A : \hat{a}(x) = 0\}$. For $\tau \in A^*$ and $a \in A$ define $a\tau$ in A^* by $a\tau(b) = \tau(ab)$ and define the support of τ by

$$\operatorname{supp}(\tau) = \{x \in X_A : a\tau \neq 0 \text{ whenever } a(x) \neq 0\}.$$

Then, for a closed set $E \subset X_A$

$$J_A(E)^{\perp} = \{\tau \in A^* : \operatorname{supp}(\tau) \subset E\}$$

and E is spectral for A if and only if $\tau(a) = 0$ for any $a \in A$ and $\tau \in A^*$ such that $\operatorname{supp}(\tau) \subset E \subset \operatorname{null}(a)$.

The algebra $A(G)$ is a semi-simple abelian regular Banach algebras with spectrum G. In what follows we write $I_A(E)$ for $I_{A(G)}(E)$, $J_A(E)$ for $J_{A(G)}(E)$, and $\operatorname{supp}_{VN} T$ for $\operatorname{supp}(T)$ if $T \in VN(G) = (A(G))^*$.

Now we recall some definitions and important facts on operator synthesis following [A,ShT1]. To make use of the results from [A,ShT1] we will assume in the rest of the paper that G is a second countable locally compact group or s.c.l.c group for short, and therefore is metrizable by Hewitt and Ross [HRI, 8.3].

A subset $E \subset G \times G$ is called *marginally null* (with respect to $m_G \times m_G$) if $E \subset (M \times G) \cup (G \times N)$ and $m_G(M) = m_G(N) = 0$. Two subsets E_1, E_2 are marginally equivalent ($E_1 \sim^M E_2$ or simply $E_1 \cong E_2$) if their symmetric difference is marginally null. Furthermore, $E_1 \subset^M E_2$ means that $E_1 \setminus E_2$ is marginally null, a property holds marginally almost everywhere if it holds everywhere apart of a marginally null set, and so on.

Following [ErKSh] we define a pseudo-topology on G. We call a subset E a pseudo-open if it is marginally equivalent to a countable union of measurable rectangles $A \times B$. The complements of pseudo-open sets are pseudo-closed sets.

Set $T(G) = L^2(G)\hat{\otimes}L^2(G)$, where $\hat{\otimes}$ denotes the projective tensor product. Note that in [ShT1] the notation $\Gamma(G, G)$ is used instead of $T(G)$. Every $\Psi \in T(G)$ can be identified with a function $\Psi : G \times G \to \mathbb{C}$ which admits a representation

$$\Psi(x, y) = \sum_{n=1}^{\infty} f_n(x)g_n(y), \tag{3.1}$$

where $f_n \in L^2(G)$, $g_n \in L^2(G)$ and $\sum_{n=1}^{\infty} \|f_n\|_2 \cdot \|g_n\|_2 < \infty$. Such a representation defines a function marginally almost everywhere (m.a.e.), so two functions in $T(G)$ which coincides m.a.e. are identified. The $L^2(G)\hat{\otimes}L^2(G)$-norm of Ψ is

$$\|\Psi\|_{T(G)} = \inf\left\{\sum_{n=1}^{\infty} \|f_n\|_2 \cdot \|g_n\|_2 : \Psi = \sum_{n=1}^{\infty} f_n \otimes g_n\right\}.$$

Note that a simple renormalization shows that each $\Psi \in T(G)$ admits a representation (3.1), such that $\sum_{n=1}^{\infty} \|f_n\|_2^2 \cdot \sum_{n=1}^{\infty} \|g_n\|_2^2 < \infty$ and the norm $\|\Psi\|_{T(G)}$ can be taken as the square root of the infimum of $\sum_{n=1}^{\infty} \|f_n\|_2^2 \cdot \sum_{n=1}^{\infty} \|g_n\|_2^2$ over all such representations. By Erdos et al. [ErKSh] any $\Psi \in T(G)$ is pseudo-continuous. Thus, if Ψ vanishes (m.a.e.) on $K \subset G$ it vanishes on a pseudo-closed set. For $\mathcal{F} \subset T(G)$, the null set, null $\mathcal{F}$, is defined to be the largest pseudo-closed set, such that each function $F \in \mathcal{F}$ vanishes on it. For a pseudo-closed set $E \subset G \times G$, let

$$\Phi(E) = \{w \in T(G) : w = 0 \text{ m.a.e. on } E\},$$

$$\Phi_0(E) = \overline{\{w \in T(G) : w = 0 \text{ on a pseudo-neighborhood of } E\}}.$$

The spaces $\Phi(E)$, $\Phi_0(E)$ are $L^\infty(G) \times L^\infty(G)$-bimodules: if $f, g \in L^\infty(G)$ and $w \in \Phi(E)$ ($w \in \Phi_0(E)$) then $f(x)g(y)w(x,y) \in \Phi(E)$ ($f(x)g(y)w(x,y) \in \Phi_0(E)$, respectively). Moreover, $\Phi(E)$ and $\Phi_0(E)$ are the largest and the smallest $L^\infty(G) \times L^\infty(G)$-invariant subspaces of $T(G)$ whose null set is E.

A subset $E \subset G \times G$ is called a *set of* (*operator*) *synthesis* or *synthetic* with respect to m_G if $\Phi(E) = \Phi_0(E)$.

It is know that $B(L^2(G)) \simeq T(G)^*$ (see [A]). The duality is given by

$$\langle T, \Psi \rangle = \sum_{n=1}^{\infty} (T f_n, \bar{g}_n)$$

for $T \in B(L^2(G))$ and $\Psi = \sum_{n=1}^{\infty} f_n \otimes g_n \in T(G)$.

Let P_U denote the multiplication operator by the characteristic function of a subset $U \subset G$. We say that $T \in B(L^2(G))$ is *supported* in $E \subset G \times G$ (or E supports T) if $P_V T P_U = 0$ for each pair of Borel sets $U \subset G$, $V \subset G$ such that $(U \times V) \cap E \cong \emptyset$. Then there exists the smallest (up to a marginally null set) pseudo-closed set, $\operatorname{supp} T$, which supports T. We write $\operatorname{supp}(T) \subset E$ if T is supported by E. In the seminal paper [A] Arveson defined a support in a similar way but using closed sets instead of pseudo-closed. This closed support, $\operatorname{supp}_A T$ can be strictly larger than $\operatorname{supp}(T)$. Then E is a set of operator synthesis if $\langle T, w \rangle = 0$ for any $T \in B(L^2(G))$ and $w \in T(G)$ with $\operatorname{supp}(T) \subset E \subset \operatorname{null} w$ (the inclusions up to a marginally null set).

We consider also the space $V^\infty(G)$ of all (marginal equivalence classes of) functions $\Psi(x, y)$ that can be written in the form (3.1) with $f_n \in L^\infty(G)$, $g_n \in L^\infty(G)$ and

$$\sum_{n=1}^{\infty} |f_n(x)|^2 \leqslant C, \quad x \in G, \quad \sum_{n=1}^{\infty} |g_n(y)|^2 \leqslant C, \quad y \in G$$

and for such Ψ, we have

$$\|\Psi\|_{V^\infty} = \inf \left\{ \left\| \sum_{n=1}^{\infty} |f_n|^2 \right\|_\infty^{1/2} \left\| \sum_{n=1}^{\infty} |g_n|^2 \right\|_\infty^{1/2} : \Psi = \sum_{n=1}^{\infty} f_n \otimes g_n \right\}.$$

In tensor notations $V^{\infty}(G) = L^{\infty}(G)\hat{\otimes}^{w^*h}L^{\infty}(G)$, the weak*-Haagerup tensor product [BSm]. Let

$$V^{\infty}_{\mathrm{inv}}(G)=\{w \in V^{\infty}(G): w(sr,tr)=w(s,t) \text{ for all } r in G \text{ and m.a.e. } (s,t)\in G\times G\}.$$

In [S], Spronk found a connection between $V^{\infty}_{\mathrm{inv}}(G)$ and the algebra $M_{\mathrm{cb}}A(G)$ of completely bounded multipliers of $A(G)$. For a function $u : G \to \mathbb{C}$ and t, $s \in G$ define

$$(Nu)(t,s) = u(ts^{-1}).$$

Theorem 3.1 (*Spronk [S]*). *The map $u \mapsto Nu$ is a complete isometry from $M_{\mathrm{cb}}A(G)$ onto $V^{\infty}_{\mathrm{inv}}(G)$.*

4. Spectral synthesis and operator synthesis

In this section, we will prove our main result establishing a connection between operator synthesis and spectral synthesis for $A(G)$, where G is a second countable locally compact group. The proofs are inspirited by the proof of Spronk and Turowska [ST, Theorem 4.6].

The Banach space $B(L^2(G))$ is a left $V^{\infty}(G)$-module with the action defined for $w = \sum_{i=1}^{\infty} \varphi_i \otimes \psi_i \in V^{\infty}(G)$ and $T \in B(L^2(G))$ by

$$w \cdot T = \sum_{i=1}^{\infty} M_{\psi_i} T M_{\varphi_i},$$

where the partial sums converge strongly. The operator $T \mapsto \sum_{i=1}^{\infty} M_{\psi_i} T M_{\varphi_i}$ we will also denote by Δ_w.

For a closed subset $E \subset G$ we set

$$E^* = \{(s,t) \in G \times G \mid st^{-1} \in E\}.$$

Lemma 4.1. *Let $S \in VN(G)$. Then*

$$\mathrm{supp}(S) \subset \{\mathrm{supp}_{VN}\, S\}^*.$$

Proof. Let U, V be closed subsets of G, such that $(U \times V) \cap \{\mathrm{supp}_{VN}(S)\}^* = \emptyset$. Then there exists an open neighborhood, W, of $\{\mathrm{supp}_{VN}\, S\}^*$, such that $(U \times V) \cap W = \emptyset$. Take f and g in $L^2(G)$, such that $\mathrm{supp}(f) \subset U$, $\mathrm{supp}(g) \subset V$. For $u = \langle \lambda(\cdot) f, g\rangle$, we have $(Sf, g) = \langle S, u\rangle$. Moreover, $\mathrm{supp}(u) \subset \overline{UV^{-1}}$ and $\mathrm{supp}(u) \cap \mathrm{supp}_{VN}\, S \subset \overline{UV^{-1}} \cap \mathrm{supp}_{VN}\, S = \emptyset$. Thus $0 = \langle S, u\rangle = (Sf, g)$. As f and g are chosen arbitrarily, $P_V S P_U = 0$. By the regularity of m_G, the last holds for any Borel sets U, V giving the statement. □

Remark 4.2. Let H be a closed subgroup of G. Then

$$H^* = \{(s,t) \in G \times G : st^{-1} \in H\} = \{(s,t) \in G \times G : Hs = Ht\}$$
$$= \{(s,t) \in G \times G : f(t) = f(s)\},$$

where $f : G \to H\backslash G$ is a continuous mapping defined by $f(t) = Ht$.

Assume $\mathrm{supp}_{VN}(S) \subset H$. By Lemma 4.1, $\mathrm{supp}(S) \subset H^*$. As for any Borel set $\Delta \subset H\backslash G$ and $\alpha = f^{-1}(\Delta)$, we have $(\alpha^c \times \alpha) \cap H^* = \emptyset$, it gives $P_{\alpha^c} S P_\alpha = 0$. Since this is true for any Δ we have also $P_\alpha S P_{\alpha^c} = 0$ implying that $P_\alpha S = S P_\alpha$ and hence S belongs to the commutant $\mathcal{B}'$ of the von Neumann algebra $\mathcal{B}$ generated by multiplication operators by functions $\varphi \in L^\infty(G)$ which are constant on the right cosets.

Theorem 4.3. *Let G be a s.c.l.c. group and $E \subset G$ be a closed subset. If E^* is synthetic with respect to m_G then E is a set of local spectral synthesis for $A(G)$.*

Proof. Assume that E^* is synthetic with respect to m_G. Let $u \in I_A^c(E)$ and $S \in VN(G)$, $\mathrm{supp}_{VN}(S) \subset E$. By Theorem 3.1, $Nu \in V^\infty(G)$. Moreover,

$$uT = \Delta_{Nu} T \qquad \text{for any } T \in VN(G). \tag{4.1}$$

In fact, if $Nu(s,t) = \sum_i \varphi_i(t)\psi_i(s)$, then

$$\Delta_{Nu}\lambda(s)f(t) = \sum_i M_{\varphi_i}\lambda(s)M_{\psi_i} f(t)$$
$$= \sum_i \varphi_i(t)\psi_i(s^{-1}t)f(s^{-1}t) = Nu(t, s^{-1}t)f(s^{-1}t) = u(s)\lambda(s)f(t)$$

for any $f \in L^2(G)$ and $s \in G$. The operator Δ_{Nu} is weakly continuous. In fact, if $S_k \to 0$ weakly, $\|S_k\| \leqslant C$ for some constant C and

$$|(\Delta_{Nu}(S_k)f, g)| \leqslant \sum_{i=1}^n |(S_k\psi_i f, \bar{\varphi}_i g)| + \sum_{i=n+1}^\infty |(S_k\psi_i f, \bar{\varphi}_i g)|$$
$$\leqslant \sum_{i=1}^n |(S_k\psi_i f, \bar{\varphi}_i g)| + \|S_k\| \left(\sum_{i=n+1}^\infty \|\psi_i f\|^2\right)^{1/2} \left(\sum_{i=n+1}^\infty \|\varphi_i g\|^2\right)^{1/2}$$
$$= \sum_{i=1}^n |(S_k\psi_i f, \bar{\varphi}_i g)| + C\left(\int_G \sum_{i=n+1}^\infty |\psi_i(t)f(t)|^2\,dt\right)^{1/2}$$
$$\times \left(\int_G \sum_{i=n+1}^\infty |\varphi_i(t)g(t)|^2\,dt\right)^{1/2}.$$

For given $\varepsilon > 0$, by Lebesgue's theorem, there exists n, such that the second summand is less than ε and then, as $S_k \to 0$ weakly, there exists K, such that the first summand is less than ε for any $k \geqslant K$. Therefore (4.1) holds for any $T \in VN(G)$.

Clearly, since $u \in I_A(E)$, Nu vanishes on E^*. By Lemma 4.1, we also have that $\text{supp}(S) \subset \{\text{supp}_{VN} S\}^* \subset E^*$. Therefore, for each $w \in T(G)$,

$$\langle \Delta_{Nu} S, w\rangle = \langle S, (Nu)w\rangle = 0,$$

so that $uS = \Delta_{Nu} S = 0$.

From the regularity of $A(G)$ it follows that there exists a compactly supported function $v \in A(G)$, such that $v = 1$ on the support of u. Thus

$$\langle S, u\rangle = \langle S, vu\rangle = \langle uS, v\rangle = 0. \qquad \square$$

It is easy to see that the condition for E to be a set of local spectral synthesis for $A(G)$ is equivalent to the condition $uS = 0$ for any $u \in A(G)$ and $S \in VN(G)$, such that $\text{supp}_{VN}(S) \subset E \subset \text{null } u$. If G is amenable then we have the implication

$$uT = 0 \Rightarrow \langle T, u\rangle = 0, \tag{4.2}$$

guaranteed by the existence of a bounded approximate identity and any set of local synthesis is a set of synthesis. Certainly the assumption of boundedness of the identity is superfluous, and the statement holds even for G, such that $A(G)$ has an unbounded approximate identity. So that we have

Corollary 4.4. *Let G be a s.c.l.c. group such that $A(G)$ has an* (*unbounded*) *approximate identity and let E be a closed subset of G. If E^* is a set of operator synthesis with respect to m_G then E is a set of spectral synthesis.*

Remark 4.5. (1) It is not known whether an approximate identity exists for every locally compact group G. Unbounded approximate identities which are completely bounded as multipliers of the Fourier algebra $A(G)$ were studied in [CaHaa,Ca,CoHaa]. Those exist for a number of groups G like the general Lorentz group $SO_0(n, 1)$, its closed subgroups, in particular, the free group $\mathbb{F}_n$ on n generators, extensions of $SO_0(n, 1)$ by a finite group, in particular, $SL(2, \mathbb{R})$, $SL(2, \mathbb{C})$, $SL(2, \mathbb{H})$. A group G, such that $A(G)$ has an approximate identity which is bounded in the completely bounded norm, is called weakly amenable. It was discovered in [Haa,HaaKr] that there exist groups which fail to be weakly amenable but still have an approximate identity (not bounded even in the multiplier norm), as e.g. $G = \mathbb{Z}^2 \times_\rho SL(2, \mathbb{Z})$, where ρ is the standard action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$.

(2) The property (4.2) of an operator $T \in VN(G)$ was discussed in [E] and called there by the property (H). In particular, it was shown that any T supported in a compact set E possesses this property. Thus any compact set of local synthesis is a set of

spectral synthesis. It is not known whether there exists $T \in VN(G)$ which does not satisfy (4.2).

Corollary 4.6. *Let E be a compact subset of G. If E^* is a set of synthesis with respect to m_G then E is a set of spectral synthesis.*

The next statement was proved by Eymard for arbitrary locally compact groups G, [E], using more complicated arguments of the theory of distributions on G.

Corollary 4.7. *Let s be an element of the s.c.l.c. group G. Then $\{s\}$ is a set of spectral synthesis.*

Proof. It is enough to prove the statement for $E = \{e\}$, where e is the identity element in G. In this case $E^* = \{(s, s) : s \in G\}$. That E^* is a set of operator synthesis follows, e.g. from [ShT1, Theorem 4.8], but it can be easily seen also using the following simple arguments.

Let $T \in B(L^2(G))$ and $\mathrm{supp}(T) \subset E^*$. It follows from Remark 4.2 that T is the multiplication operator by some function $a \in L^\infty(G)$. Let now $F = \sum_{i=1}^{\infty} f_i \otimes g_i \in \Phi(E^*)$. Then

$$\langle T, F\rangle = \sum_{i=1}^{\infty} (Tg_i, \bar{f}_i) = \sum_{i=1}^{\infty} (ag_i, \bar{f}_i) = \int_G a(r)F(r, r)\,dr = 0,$$

and therefore E^* is a set of synthesis with respect to m_G. By Corollary 4.6, E is a set of spectral synthesis. □

We also have another proof in the case of s.c.l.c. groups of the following statement proved by Herz [He, Theorem 2].

Corollary 4.8. *Any closed subgroup H of G is a set of local spectral synthesis.*

Proof. We have $H^* = \{(s, t) \in G \times G : st^{-1} \in H\} = \{(s, t) \in G \times G : Hs = Ht\} = \{(s, t) \in G \times G : f(t) = f(s)\}$, where $f : G \to H\backslash G$ is a continuous function defined by $f(t) = Ht$. By Shulman and Turowska [ShT1, Theorem 4.8], H^* is a set of operator synthesis and the statement now follows from Theorem 4.3. □

Remark 4.9. By Corollaries 4.4, 4.6, any compact subgroup H is spectral for $A(G)$ and any closed subgroup is spectral for $A(G)$ if $A(G)$ has an approximate identity.

That any closed subgroup of a locally compact group G is a set of spectral synthesis was shown by Takesaki and Tatsuuma [TT, Theorem 3], using the Mackey imprimitivity theorem and the result mentioned in Remark 4.2 proved by a different method.

Example 4.10. Let R be an ordered s.c.l.c. group and $\Delta : G \to R$ be a continuous homomorphism. Take a finite intersection of intervals $\cap_{k=1}^{m} [\alpha_k, \beta_k]$ in R and set $E = \Delta^{-1}(\cap_{k=1}^{m}[\alpha_k, \beta_k])$. Then

$$\begin{aligned} E^* &= \{(s,t) : \alpha_k \leqslant \Delta(st^{-1}) \leqslant \beta_k, k = 1, \ldots, m\} \\ &= \{(s,t) : \alpha_k \Delta(t) \leqslant \Delta(s) \leqslant \beta_k \Delta(t), k = 1, \ldots, m\} \\ &= \{(s,t) : f_i^k(t) \leqslant g_i^k(s), i = 1, 2, k = 1, \ldots, m\}, \end{aligned}$$

where $f_1^k(t) = \alpha_k \Delta(t)$, $f_2^k(t) = \Delta(t)^{-1}$, $g_1^k(s) = \Delta(s)$, $g_2^k(s) = \Delta(s)^{-1}\beta_k$. By Shulman and Turowska [ShT1, Theorem 4.8], E^* is a set of operator synthesis with respect to the Haar measure on G. Therefore E is a set of spectral synthesis by Theorem 4.3.

Our next aim is to prove a converse to the statement of Theorem 4.3.

The Banach space $T(G)$ is an $L^1(G)$-module with the action defined by

$$f \odot w(s,t) = \int_G f(r)\Delta^{1/2}(r) w(sr, tr)\, dr, \quad f \in L^1(G), w \in T(G),$$

where Δ is the modular function of G. We have $\|f \odot w\|_{T(G)} \leqslant \|f\|_1 \cdot \|w\|_{T(G)}$. Moreover, $e_\alpha \odot w \to w$ for any bounded approximate identity $\{e_\alpha\}$ in $L^1(G)$ (see [HRII, 32.22, 32.33] and [ST, p. 365]).

Let us define another action by compactly supported $L^1(G)$-functions f:

$$f \cdot w(s,t) = \int_G f(r) w(sr, tr)\, dr, \quad w \in T(G).$$

Then $f \odot w = f\Delta^{1/2} \cdot w$.

We observe that by the estimate (4.3) below, the integral $\int_G f(r)w(sr,tr)\,dr$, $w \in T(G)$, $s,t \in G$, converges also if $f \in L^\infty(G)$ and so defines a mapping $(s,t) \to f \cdot w(s,t) := \int_G f(r) w(sr,tr)\,dr$.

Theorem 4.11. *Let G be a s.c.l.c. group. If a closed subset $E \subset G$ is a set of local spectral synthesis for $A(G)$ then E^* is synthetic with respect to Haar measure.*

Proof. It is sufficient to show that $w \cdot T = 0$ for $T \in B(L^2(G))$ and $w \in V^\infty(G)$ such that $\operatorname{supp}(T) \subset E^* \subset \operatorname{null} w$ [ShT2, Proposition 5.3].

As G is second countable the group G is σ-compact and therefore there exist compact sets K_n, such that $K_n \subset K_{n+1}$ and $\cup_{n=1}^{\infty} K_n = G$. Then clearly, $M_{\chi_{K_n}} T M_{\chi_{K_n}} \to T$ strongly. Therefore we can restrict ourselves to a compactly supported operator T, i.e. $\operatorname{supp}(T) \subset M \times M$ for a compact set $M \subset G$, and compactly supported $w \in V^\infty(G)$. Note that in this case $w \in T(G)$.

Let $\hat{G}$ denote the set of (equivalence classes of) irreducible continuous unitary representations of G. For $\pi \in \hat{G}$ let H_π denote the representation space of π. Fix a basis $\{e_j\}_j$ in H_π and denote by u_{kj}^π the matrix coefficients of π, i.e. $u_{kj}^\pi(s) = (\pi(s)e_k, e_j)$.

For (a compactly supported) $w = \sum_{i=1}^{\infty} \varphi_i \otimes \psi_i \in T(G)$ and $\pi \in \hat{G}$ consider now the following operator-valued function

$$w^{\pi}(s,t) = \int_G w(sr,tr)\pi(r)\,dr.$$

The integral is well-defined as a Bochner integral. In fact, for each s, $t \in G$, applying Cauchy–Schwartz's inequality, we obtain

$$\begin{aligned}
\int_G |w(sr,tr)|\,dr &\leqslant \int_G \sum_{i=1}^{\infty} |\varphi_i(sr)\psi_i(tr)|\,dr \\
&\leqslant \int_G \left(\sum_{i=1}^{\infty} |\varphi_i(sr)|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |\psi_i(tr)|^2\right)^{1/2} dr \\
&\leqslant \left(\sum_{i=1}^{\infty} \int_G |\varphi_i(sr)|^2\,dr\right)^{1/2} \left(\sum_{i=1}^{\infty} \int_G |\psi_i(tr)|^2\,dr\right)^{1/2} \\
&= \left(\sum_{i=1}^{\infty} \|\varphi_i\|_2^2\right)^{1/2} \left(\sum_{i=1}^{\infty} \|\psi_i\|_2^2\right)^{1/2} < \infty.
\end{aligned} \tag{4.3}$$

Set $\tilde{w}^{\pi}(s,t) = \pi(s)w^{\pi}(s,t)$ and

$$w_{kj}^{\pi}(s,t) = (w^{\pi}(s,t)e_k, e_j) = u_{kj}^{\pi} \cdot w(s,t), \quad \tilde{w}_{kj}^{\pi}(s,t) = (\tilde{w}^{\pi}(s,t)e_k, e_j). \tag{4.4}$$

If $w \in \Phi(E^*)$ then $w(sr,tr)$ vanishes m.a.e. on E^* for all r, and therefore $w^{\pi}(s,t)$, $\tilde{w}^{\pi}(s,t)$, $w_{kj}^{\pi}(s,t)$ and $\tilde{w}_{kj}^{\pi}(s,t)$ vanish on E^*.

We have the following expression for w_{kj}^{π} and $\tilde{w}_{kj}^{\pi}$:

$$\begin{aligned}
\tilde{w}_{kj}^{\pi}(s,t) = \int_G w(sr,tr)(\pi(sr)e_k,e_j)\,dr &= \int_G w(r,ts^{-1}r)(\pi(r)e_k,e_j)\,dr \\
&= \int_G w(r,ts^{-1}r)u_{kj}^{\pi}(r)\,dr.
\end{aligned}$$

In particular, if $w = f \otimes g \in T(G)$, then

$$\tilde{w}_{kj}^{\pi}(s,t) = \int_G f(r)u_{kj}^{\pi}(r)g(ts^{-1}r)\,dr = N(fu_{kj}^{\pi} * \check{g})(s,t). \tag{4.5}$$

Furthermore

$$
\begin{aligned}
w_{kj}^{\pi}(s,t) &= \int_G w(sr,tr)(\pi(r)e_k,e_j)\,dr = \int_G w(r,ts^{-1}r)(\pi(s^{-1}r)e_k,e_j)\,dr \\
&= \int_G w(r,ts^{-1}r)(\pi(r)e_k,\pi(s)e_j)\,dr \\
&= \sum_l \int_G w(r,ts^{-1}r)(\pi(r)e_k,e_l)(e_l,\pi(s)e_j)\,dr \\
&= \sum_l u_{lj}^{\pi}(s)\int_G w(r,ts^{-1}r)u_{kl}^{\pi}(r)\,dr = \sum_l u_{lj}^{\pi}(s)\tilde{w}_{kl}^{\pi}(s,t). \qquad (4.6)
\end{aligned}
$$

We state first that $\tilde{w}_{kj}^{\pi}(s,t)\in V^{\infty}(G)$. Indeed, if $w=f\otimes g\in T(G)$, then by (4.5) $\tilde{w}_{kj}^{\pi}=N(fu_{kj}^{\pi}*\check{g})$ and therefore, since $fu_{kj}^{\pi}*\check{g}\in A(G)$, $\tilde{w}_{kj}^{\pi}\in V^{\infty}(G)$, by Theorem 3.1. Moreover,

$$
\|\tilde{w}_{kj}^{\pi}(s,t)\|_{V^{\infty}}\leqslant\|fu_{kj}\|_2\|g\|_2\leqslant\|f\|_2\|g\|_2,
$$

so that the linear operator $w\mapsto\tilde{w}_{kj}^{\pi}\in V^{\infty}(G)$ defined on elementary tensors extends to a bounded operator $T(G)\to V^{\infty}(G)$. Thus $\tilde{w}_{kj}^{\pi}\in V^{\infty}(G)$ for any $w\in T(G)$.

Next we show that $w_{kj}^{\pi}\in V^{\infty}(G)$. For $T\in B(L^2(G))$ and $w=f\otimes g\in T(G)$ such that $\|w\|=\|f\|_2\|g\|_2$, define $w_{kj}^{\pi}\cdot T$ by

$$
\langle w_{kj}^{\pi}\cdot T,\Psi\rangle=\sum_l\langle\tilde{w}_{kl}^{\pi}\cdot T,u_{lj}^{\pi}(s)\Psi(s,t)\rangle,\quad \Psi\in T(G). \qquad (4.7)
$$

This formula makes sense. In fact, if $\Psi=\sum_{i=1}^{\infty}f_i\otimes g_i$, such that $\|\Psi\|_{T(G)}^2=\sum_{i=1}^{\infty}\|f_i\|^2\sum_{i=1}^{\infty}\|g_i\|^2$, we have

$$
\begin{aligned}
\left|\sum_l\langle\tilde{w}_{kl}^{\pi}\cdot T,u_{lj}^{\pi}\Psi\rangle\right|^2 &\leqslant\left(\sum_l\|\tilde{w}_{kl}^{\pi}\cdot T\|^2\right)\left(\sum_l\|u_{lj}^{\pi}\Psi\|_{T(G)}^2\right) \\
&\leqslant\left(\sum_l\|\tilde{w}_{kl}^{\pi}\|_{V^{\infty}}^2\|T\|^2\right)\left(\sum_l\sum_{i=1}^{\infty}\|u_{lj}^{\pi}f_i\|_2^2\cdot\sum_{i=1}^{\infty}\|g_i\|_2^2\right) \\
&\leqslant\|T\|^2\left(\sum_l\|fu_{kl}^{\pi}\|_2^2\|g\|_2^2\right)\left(\sum_{i=1}^{\infty}\sum_l\|u_{lj}^{\pi}f_i\|_2^2\right)\cdot\sum_{i=1}^{\infty}\|g_i\|_2^2 \\
&=\|T\|^2\|f\|_2^2\|g\|_2^2\sum_{i=1}^{\infty}\|f_i\|_2^2\cdot\sum_{i=1}^{\infty}||g_i\|_2^2 \\
&\leqslant\|T\|^2\|w\|_{T(G)}^2\|\Psi\|_{T(G)}^2,
\end{aligned}
$$

the last equality follows from

$$\sum_l \|fu_{kl}\|_2^2 = \sum_l \int_G |f(t)u^\pi_{kl}(t)|^2\,dt = \int_G |f(t)|^2 \sum_l |(\pi(t)e_k, e_l)|^2\,dt$$
$$= \int_G |f(t)|^2 \|(\pi(t)e_k\|_2^2\,dt = \int_G |f(t)|^2\,dt = \|f\|_2^2.$$

Thus, the operator $w^\pi_{kj}\cdot T$ is well-defined for $w=f\otimes g$ and since $\|w^\pi_{kj}\cdot T\| \leqslant \|T\|\|w\|_{T(G)}$ for elementary tensors w, the definition $w^\pi_{kj}\cdot T$ makes sense for any $w\in T(G)$ and

$$\left|\sum_l \langle \tilde{w}^\pi_{kl}\cdot T, u^\pi_{lj}\Psi\rangle\right|^2 \leqslant \|T\|^2\|w\|^2_{T(G)}\|\Psi\|^2_{T(G)} \tag{4.8}$$

for any $\Psi\in T(G)$. Clearly, $T\mapsto w^\pi_{kj}\cdot T$ is thus a bounded $L^\infty(G)$-bimodule map on $B(L^2(G))$ and hence by Smith [Sm, 2.1] is completely bounded. Then by Blecher and Smith [BSm, 4.2], it is of the form $\omega\cdot T$ for $\omega\in V^\infty(G)$ and therefore $w^\pi_{kj}=\omega$ m.a.e.

For $\tilde{w}^\pi$ we have

$$\tilde{w}^\pi(sr,tr) = \pi(sr)w^\pi(sr,tr) = \pi(sr)\int_G w(srp,trp)\pi(p)\,dp$$
$$= \pi(s)\int_G w(srp,trp)\pi(rp)\,dp = \pi(s)\int_G w(sp,tp)\pi(p)\,dp = \tilde{w}^\pi(s,t),$$

implying $\tilde{w}^\pi_{kj}\in V^\infty_{\mathrm{inv}}(G)$. By Theorem 3.1,

$$\tilde{w}^\pi_{kj} = Nu \tag{4.9}$$

for some $u\in M_{\mathrm{cb}}A(G)$. Moreover, if w vanishes on E^* then u vanishes on E.

We claim that $Nu\cdot T=0$ for any operator T and $u\in M_{\mathrm{cb}}A(G)$ such that $\mathrm{supp}(T)\subset E^*$, null $u\supset E$. In fact, since E is a set of local spectral synthesis, given $w\in A(G)$ with compact support, uw can be approximated by $u_\alpha\in J^0_A(E)$ and therefore $N(uw)$ is a $V^\infty(G)$-limit of Nu_α, vanishing on pseudo-neighborhoods of E^*. By Shulman and Turowska [ShT1, Theorem 4.3], $\langle T,\Psi\rangle = 0$ for any $\Psi\in\Phi_0(E^*)$. Therefore $\langle Nu_\alpha\cdot T, F\rangle = \langle T,(Nu_\alpha)F\rangle = 0$ for any $F\in T(G)$ implying $Nu_\alpha\cdot T=0$ and $N(uw)\cdot T=0$. As $\langle Nu\cdot T,(Nw)F\rangle=0$ for any $w\in A(G)$ with compact support and $F\in T(G)$ it is enough to see now that the subspace, $\mathcal{M}$, generated by $(Nw)F$, where $F\in T(G)$ and $w\in A(G)$ with compact support, is dense in $T(G)$. As null $\mathcal{M}=\emptyset$, this is true by an analogue of Wiener's Tauberian Theorem [ShT1, Corollary 4.3].

We obtain by (4.9) that for any $w\in V^\infty(G)$ which is compactly supported and vanishes on E^*

$$\tilde{w}^\pi_{kj}\cdot T = 0.$$

Therefore by (4.8)

$$w_{kj}^{\pi} \cdot T = \left(\sum_{l} u_{jl}^{\pi}(s) \tilde{w}_{kj}^{\pi}(s,t) \right) \cdot T = 0,$$

Let K be a compact set such that $\operatorname{supp}(w) \subset K \times K$ and $\operatorname{supp}(T) \subset K \times K$. Then since $T = M_{\chi_K} T M_{\chi_K}$, for compactly supported $h \in L^1(G)$ we have

$$\langle T, h \cdot w \rangle = \langle M_{\chi_K} T M_{\chi_K}, h \cdot w \rangle = \left\langle T, \chi_K(t)\chi_K(s) \int_G w(sr, tr) h(r)\, dr \right\rangle$$
$$= \left\langle T, \int_G w(sr, tr) \chi_{K^{-1}K}(r) h(r)\, dr \right\rangle = \langle T, h\chi_{K^{-1}K} \cdot w \rangle.$$

Take $f \in L^{\infty}(G)$, $g \in L^{\infty}(G)$, $\operatorname{supp}(f) \subset K$, $\operatorname{supp}(g) \subset K$, and set $\omega = f \otimes g$. Then

$$\langle \omega \cdot T, u_{kj}^{\pi} \chi_{K^{-1}K} \cdot w \rangle = \langle T, (u_{kj}^{\pi} \chi_{K^{-1}K} \cdot w)\omega \rangle$$
$$= \langle T, w_{kj}^{\pi} \omega \rangle = \langle w_{kj}^{\pi} \cdot T, \omega \rangle = 0.$$

Hence for finite linear combinations $\sum_i c_i u_i$ of matrix coefficients

$$\left\langle \omega \cdot T, \sum_i c_i u_i \chi_{K^{-1}K} \cdot w \right\rangle = 0.$$

Take now an approximate identity $\{e_\alpha\}$ consisting of non-negative continuous functions with compact support in $L^1(G)$. We can assume that $K^{-1}K$ contains the supports of e_α's. Then, by Dixmier [Di, 13.6.5], $\Delta^{1/2} e_\alpha$ can be approximated in $L^1(G)$ by finite linear combinations $\sum_{i=1}^{d} c_i u_i \chi_{K^{-1}K}$. This yields

$$0 = \langle \omega \cdot T, (\Delta^{1/2} e_\alpha) \cdot w \rangle = \langle \omega \cdot T, (e_\alpha) \odot w \rangle$$

and therefore $0 = \langle \omega \cdot T, w \rangle = \langle w \cdot T, \omega \rangle$. Finally, we obtain $w \cdot T = 0$. □

Corollary 4.12. *Let G be a s.c.l.c. group.*

(a) *Then a compact set $E \subset G$ is a set of spectral synthesis for $A(G)$ if and only if E^* is a set of operator synthesis with respect to Haar measure.*
(b) *Assume that $A(G)$ has an approximate identity. Then a closed set $E \subset G$ is a set of spectral synthesis for $A(G)$ if and only if E^* is a set of operator synthesis with respect to Haar measure.*

Proof. Follows from Corollaries 4.4, 4.6 and Theorem 4.11. □

5. Ditkin sets and operator Ditkin sets

Our goal in this section is to find a connection between Ditkin sets and operator Ditkin sets. Let G be a locally compact group and let m_G be the Haar measure on G. A closed subset $E \subset G$ is said to be a (local) Ditkin set if for any $f \in I_A(E)$ ($f \in I_A^c(E)$) there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \in J_A^0(E)$ ($n = 1, 2, \ldots$) and $\tau_n f \to f$ as $n \to \infty$. It is called a strong Ditkin set if such a sequence $\{\tau_n\}$ can be chosen uniformly for all functions $f \in I_A(E)$.

In a similar way operator Ditkin sets were defined in [ShT1]. By Spronk and Turowska [ST] the elements of $V^{\infty}(G)$ are the multipliers of $T(G)$, i.e. $w\omega \subset \omega$ if $w \in V^{\infty}(G)$ and $\omega \in T(G)$. We call a pseudo-closed subset $E \subset G \times G$ an m_G-Ditkin set if for any $w \in \Phi(E)$ there exists a sequence $\tau_n \in V^{\infty}(G)$ such that τ_n vanishes on a pseudo-neighborhood of E ($n = 1, 2, \ldots$) and

$$\|\tau_n w - w\|_{T(G)} \to 0, \quad \text{as } n \to \infty.$$

$E \subset G \times G$ is said to be a strong m_G-Ditkin set if such a sequence $\{\tau_n\}_{n=1}^{\infty}$ can be chosen uniformly for all $w \in T(G)$.

If G is a compact metrizable abelian group, it is known that $E = \{0_G\}$ is a strong Ditkin set. Moreover, this set satisfies the following three conditions: (1) E is a set of synthesis; (2) there exist open sets Ω_n containing E, such that

$$\Omega_{n+1} \subset \Omega_n \quad n = 1, 2, \text{ and } \cap_{n=1}^{\infty} \bar{\Omega}_n = E;$$

(3) there exists a sequence $\{u_n\}$ with $1 - u_n \in J_A^0(E)$, $n = 1, 2, \ldots$, satisfying the following two conditions:

$$u_n(x) = 0 \quad \text{for all } x \notin \Omega_n,$$

$$\|u_n\| \leqslant 1 + \varepsilon_n,$$

where $\{\varepsilon_n\}$ is a sequence decreasing to zero.

Note that any closed subset E of the spectrum of a semisimple, regular, commutative Banach algebra A satisfying the conditions (1)–(3) is a strong Ditkin set for A [D].

Let Z be a standard Borel space and let $f : G \to Z$ and $g : G \to Z$ be Borel functions. Consider

$$E = \{(s, t) : f(s) = g(t)\} \subset G \times G. \tag{5.1}$$

If G, Z are compact metrizable spaces, f, g are continuous functions and $A = V(G) = C(G)\hat{\otimes}C(G)$, the Varopoulos algebra, then E is shown in [D] to satisfy the conditions (1)–(3) and therefore to be a strong Ditkin set for $V(G)$. Knowing that E is a set of synthesis with respect to m_G [ShT1, Theorem 4.8] we can use a similar argument to show the following statement.

Proposition 5.1. *$E = \{(s,t) : f(s) = g(t)\} \subset G \times G$ is a strong m_G-Ditkin set.*

Proof. First we embed Z into the torus, $\mathbb{T}_\infty$, of infinite dimension: by Takesaki [T, Theorem A.1] there exists a Borel injective mapping $\psi : Z \to \mathbb{T}_\infty$. Consider a mapping $\rho : G \times G \to \mathbb{T}_\infty$ given by

$$\rho(s,t) = \psi(f(t)) - \psi(g(s)),$$

where the subtraction is taken with respect to the group structure on $\mathbb{T}_\infty$. Then $\rho^{-1}(\mathbb{T}_\infty) = E$.

As $\{0_{\mathbb{T}_\infty}\}$ satisfies (1)–(3) in $A(\mathbb{T}_\infty)$, there exist open sets $\Sigma_n \subset \mathbb{T}_\infty$, such that $0_{\mathbb{T}_\infty} \in \Sigma_n$, $\Sigma_n \subset \Sigma_{n+1}$, $n = 1, 2, \ldots, \bigcap \bar{\Sigma}_n = \{0\}$ and a sequence of functions $w_n \in A(\mathbb{T}_\infty)$, such that $1 - w_n \in J_A^0(\{0_{\mathbb{T}_\infty}\})$, $w_n(x) = 0$ for $x \notin \Sigma_n$ and $\|w_n\|_A \leqslant 1 + \varepsilon_n$ ($\varepsilon_n \to 0$, $\varepsilon_n > 0$).

Define $\Omega_n = \rho^{-1}(\Sigma_n)$. As $\psi \circ f$ and $\psi \circ g$ are Borel mappings for given $m > 0$ there exist by Lusin's theorem closed subsets $A_m \subset G$ and $B_m \subset G$, such that ρ is continuous on $A_m \times B_m$ and $|G \setminus A_m| < 1/m$, $|G \setminus B_m| < 1/m$. We can choose those subsets increasing in m. Therefore

$$\Omega_n \cap (A_m \times B_m) = \{(x,y) \in A_m \times B_m : \rho(x,y) \in \Sigma_n\}$$

is open in $A_m \times B_m$ for all n and

$$\begin{aligned} E \cap (A_m \times B_m) &= \cap_{n=1}^{\infty} \rho^{-1}(\Sigma_n) \cap (A_m \times B_m) \\ &\subset \cap_{n=1}^{\infty} \overline{\Omega_n \cap (A_m \times B_m)} \subset \\ &\subset \cap_{n=1}^{\infty} \rho^{-1}(\overline{\Sigma_n}) \cap (A_m \times B_m) = E \cap (A_m \times B_m) \end{aligned} \tag{5.2}$$

Set $u_n = w_n \circ \rho$. First we show that $u_n \in V^\infty(G)$. In fact, if $w_n = \sum_{\chi \in \hat{\mathbb{T}}_\infty} a_{\chi,n} \chi$ with $\sum_{\chi \in \hat{\mathbb{T}}_\infty} |a_{\chi,n}| = \|w_n\|_A$, then

$$w_n(\rho(s,t)) = \sum_{\chi \in \hat{\mathbb{T}}_\infty} a_{\chi,n} \chi(\psi(f(s))) \overline{\chi(\psi(g(t)))}$$

and our claim follows since

$$\sum_{\chi \in \mathbb{T}_\infty} |a_{\chi,n}| |\chi(\psi(f(s)))|^2 = \sum_{\chi \in \hat{\mathbb{T}}_\infty} |a_{\chi,n}| |\chi(\psi(g(t)))|^2 = \sum_{\chi \in \hat{\mathbb{T}}_\infty} |a_{\chi,n}| = \|w_n\|_A.$$

Moreover, by Theorem 3.1, $\|u_n\|_{V^\infty} \leqslant \|w_n\|_A \leqslant 1+\varepsilon_n$, $u_n = 0$ on Ω_n^{c} and $\tau_n = 1-u_n$ vanishes on a pseudo-neighborhood of E.

We next show that for given $w \in \Phi(E)$, $\|\tau_n w - w\| \to 0$, as $n \to 0$. Assume first that $\mathrm{supp}(w) \subset K \times K$, where K is a compact set. Then $w = w_1^m + w_2^m + w_3^m + w_4^m$, where $w_1^m = w\chi_{A_m \times B_m}$, $w_2^m = w\chi_{(G\setminus A_m)\times B_m}$ $w_3^m = w\chi_{A_m\times(G\setminus B_m)}$, $w_4^m = w\chi_{(G\setminus A_m)\times(G\setminus B_m)}$. For given $\varepsilon > 0$ there exists $M > 0$, such that $\|w_i^m\|_{T(G)} < \varepsilon$ for each $m \geqslant M$ and $i = 2, 3, 4$. Indeed, since the sequence $\{A_m\}$ is increasing in measure to G, by Lebesgue's theorem $\int_{G\setminus A_m} \sum_{i=1}^\infty |f_i(r)|^2\,dr \to 0$ as $m \to \infty$ and for w_2^m we have

$$\|w_2^m\|_{T(G)}^2 \leqslant \int_{G\setminus A_m} \sum_{i=1}^\infty |f_i(r)|^2\,dr \int_G \sum_{i=1}^\infty |g_i(r)|^2\,dr \to 0.$$

Similarly, $\|w_i^m\|_{T(G)}^2 \to 0$, $i = 3, 4$. Fix now $m > M$. By Shulman and Turowska [ShT1, Theorem 4.8] E is a set of operator synthesis with respect to Haar measure and by Shulman and Turowska [ShT1, Lemma 6.1] so is $E \cap (A_m \times B_m)$. As $E \cap (A_m \times B_m)$ is closed, by Arveson [A, 2.29] there exists $\psi \in T(G)$, $\mathrm{supp}\,\psi \subset A_m \times B_m$, vanishes on an open neighborhood $\Omega \subset A_m \times B_m$ of $E \cap (A_m \times B_m)$ and $\|w_1^m - \psi\| < \varepsilon$. By (5.2),

$$\cap_{n=1}^\infty \overline{\Omega_n \cap (A_m \times B_m)} \cap K \times K \subset \Omega \cap K \times K$$

and so $\Omega_n \cap (A_m \times B_m) \cap K \times K \subset \Omega \cap K \times K$ for $n > N$, N is large enough. Thus $u_n\psi\chi_{K\times K} = 0$ for $n > N$. We obtain

$$\|\tau_n w_1^m - w_1^m\| = \|\tau_n(w_1^m - \psi)\chi_{K\times K} - u_n\psi\chi_{K\times K} - (w_1^m-\psi)\chi_{K\times K}\| \leqslant (1+\|\tau_n\|_{V^\infty})\varepsilon$$

and

$$\|\tau_n w - w\| \leqslant \|\tau_n w_1^m - w_1^m\| + \sum_{i=2}^4 \|(\tau_n - 1)w_i^m\| \leqslant 4(1+\|\tau_n\|_{V^\infty})\varepsilon.$$

As G is σ-compact there exist compact subsets K_l, such that $K_l \subset K_{l+1}$ and $\cup_{l=1}^\infty K_l = G$, and therefore for any $w \in T(G)$, $w = \lim w\chi_{K_l\times K_l}$ in $T(G)$ giving that $\|\tau_n w - w\| \to 0$ for any $w \in \Phi(E)$. $\square$

Corollary 5.2. *Any finite union of sets of type* (5.1) *is a Ditkin set.*

Proof. It follows from Proposition 5.1 and [ShT1, Theorem 7.1]. $\square$

Remark 5.3. It follows, in particular, from Corollary 5.2 that finite unions of sets of type (5.1) are sets of operator synthesis. This was also proved by Todorov in [To] using another method.

We will now establish a connection between (strong) Ditkin sets for $A(G)$ and (strong) operator Ditkin sets.

For $w \in T(G)$, define as in [V]

$$Qw(s) = \int_G w(sr, r)\, dr.$$

If $w = \sum_{i=1}^{\infty} f_i \otimes g_i$ with $\sum_{i=1}^{\infty} \|f_i\|_2^2 \cdot \sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$. Then

$$Qw(s) = \sum_{i=1}^{\infty} (g_i * \check{f}_i)(s^{-1}) \in A(G)$$

and, moreover, $\|Qw\| \leqslant \|w\|_{T(G)}$. Thus $Q : T(G) \to A(G)$ defines a contraction operator.

Theorem 5.4. *Let G be a second countable locally compact group. If E^* is a strong m_G-Ditkin set then E is a local Ditkin set. If E is a strong Ditkin set then E^* is an m_G-Ditkin set.*

Proof. Assume first that E^* is a strong m_G-Ditkin set. Let $\{\Psi_n\}_{n=1}^{\infty} \subset V^{\infty}(G)$ be a sequence from the definition of a strong m_G-Ditkin set and let $u \in I_c(E)$. For a compact subset $K \subset G$ containing the support of u, define $(Nu)_K(s,t) = u(st^{-1})\chi_K(t)$. We have $(Nu)_K(s,t) = \chi_{MK}(s)u(st^{-1})\chi_K(t)$, where $M = \mathrm{supp}(u)$. As $u(st^{-1}) \in V^{\infty}(G)$ and $|MK| < \infty$, $|K| < \infty$, it yields $(Nu)_K \in T(G)$. Moreover, $(Nu)_K$ vanishes on E^*. Therefore, $\|\Psi_n(Nu)_K - (Nu)_K\|_{T(G)} \to 0$ as $n \to \infty$. Thus given $\varepsilon > 0$, there exists N, such that for $n > N$

$$\|Q(\Psi_n(Nu)_K) - Q((Nu)_K)\|_A \leqslant \|\Psi_n(Nu)_K - (Nu)_K\|_{T(G)} < \frac{\varepsilon}{|K|}.$$

On the other hand,

$$\begin{aligned} Q(\Psi_n(Nu)_K)(s) &= \int_G \Psi_n(sr,r)Nu(sr,r)\chi_K(r)\, dr = \int_G \Psi_n(sr,r)u(s)\chi_K(r)\, dr \\ &= u(s)\int_G \chi_{MK}(sr)\Psi_n(sr,r)\chi_K(r)\, dr = u(s)Q(\Psi_n(\chi_{MK} \otimes \chi_K))(s) \end{aligned}$$

and similarly $Q((Nu)_K) = u|K|$ giving us

$$\|Q(\Psi_n(Nu)_K) - Q((Nu)_K)\|_A = \|uw_n - u|K|\|,$$

where $w_n = Q(\Psi_n(\chi_M \otimes \chi_K))$. We obtain now that for $\tau_n = w_n/|K|$,

$$\|u\tau_n - u\|_A < \frac{|K|\varepsilon}{|K|} = \varepsilon.$$

Hence E is a local Ditkin set.

Suppose E is a strong Ditkin set. Let $v_n \in J_A^0(G)$ be a sequence, such that $\|v_n f - f\|_A \to 0$ as $n \to \infty$ for any $f \in I(E)$. By Proposition 3.1, $Nv_n \in V^\infty(G)$, $n = 1, 2, \ldots,$ and

$$\|(Nv_n)(Nf) - Nf\|_{V^\infty} \leqslant \|v_n f - f\|_A \to 0. \tag{5.3}$$

Next we show that $(Nv_n)w \cdot T \to w \cdot T$ ultra-weakly for any compactly supported T in $B(L^2(G))$, and any compactly supported w in $V^\infty(G)$, such that $w = 0$ on E^* using arguments similar to the ones in the proof of Theorem 4.11.

Assume $\mathrm{supp}(w) \subset K \times K$, for a compact set $K \subset G$. Let w_{kj}^π and $\tilde{w}_{kj}^\pi$ be V^∞-functions as defined in (4.4) and let u_{kj}^π be the matrix coefficient $(\pi(\cdot)e_k, e_j)$, $\pi \in \hat{G}$. We have $w_{kj}^\pi(s,t) = \sum_l u_{lj}^\pi(s)\tilde{w}_{kl}^\pi(s,t)$ by (4.6). Hence, for $u \in A(G)$, $\Psi \in T(G)$,

$$\begin{aligned}|\langle (Nv_n - 1)w_{kj}^\pi \cdot T, N(u)\Psi\rangle| &\leqslant \sum_l |\langle (Nv_n - 1)\tilde{w}_{kl}^\pi \cdot T, u_{jl}^\pi N(u)\Psi\rangle| \\ &= \sum_{l\in L} |\langle (Nv_n - 1)\tilde{w}_{kl}^\pi \cdot T, u_{jl}^\pi N(u)\Psi\rangle| + \sum_{l\notin L} |\langle (Nv_n - 1)\tilde{w}_{kl}^\pi \cdot T, u_{jl}^\pi N(u)\Psi\rangle|,\end{aligned}$$

where L is a finite set. For the infinite sum we can apply the estimate (4.8) with $(Nv_n - 1)\Psi$ instead of Ψ, so that for given $\varepsilon > 0$ there exists a finite subset L such that

$$\sum_{l\notin L} |\langle (Nv_n - 1)\tilde{w}_{kl}^\pi \cdot T, u_{jl}^\pi N(u)\Psi\rangle| = \sum_{l\notin L} |\langle \tilde{w}_{kl}^\pi \cdot T, u_{jl}^\pi N(u)(Nv_n - 1)\Psi\rangle| < \varepsilon.$$

For the finite sum we note first that $\tilde{w}_{kl}^\pi = N(v_{kl}^\pi)$ for some $v_{kl}^\pi \in M_{\mathrm{cb}}A(G)$ by (4.9). Then by (5.3) there exists $M > 0$, such that for any $n > M$,

$$\begin{aligned}\sum_{l\in L} |\langle (Nv_n - 1)\tilde{w}_{kl}^\pi \cdot T, u_{jl}^\pi N(u)\Psi\rangle| &= \sum_{l\in L} |\langle T, u_{jl}^\pi (Nv_n - 1)N(v_{kl}^\pi)N(u)\Psi\rangle| \\ &= \sum_{l\in L} |\langle T, u_{jl}^\pi (Nv_n - 1)N(v_{kl}^\pi u)\Psi\rangle| \\ &\leqslant \sum_{l\in L} \|T\| \cdot \|(Nv_n - 1)N(v_{kl}^\pi u)\|_{V^\infty} \|\Psi\|_{T(G)} < \varepsilon.\end{aligned}$$

We obtain,

$$\langle (Nv_n - 1)w^{\pi}_{kj} \cdot T, N(u)\Psi \rangle \to 0, \; n \to \infty.$$

As in the proof of Theorem 4.11, it yields

$$\langle (Nv_n - 1)w^{\pi}_{kj} \cdot T, \Psi \rangle \to 0, \; n \to \infty \tag{5.4}$$

for any $\Psi \in T(G)$, since $\{Nv_n\}$ is bounded in norm. Using the same arguments as in the end of the proof of Theorem 4.3 we have that for big enough compact set K there exists a linear combination $u = \sum_{i=1}^{d} c_i u_i$ of matrix coefficients such that

$$\|w - u\chi_{K^{-1}K} \cdot w\|_{T(G)} < \varepsilon.$$

Thus for $\omega \in T(G)$, $\operatorname{supp}(\omega) \subset K \times K$,

$$\begin{aligned} |\langle (Nv_n - 1)w \cdot T, \omega \rangle| &\leqslant |\langle (Nv_n - 1) \cdot T, \omega(w - u\chi_{K^{-1}K} \cdot w) \rangle| \\ &\quad + |\langle (Nv_n - 1) \cdot T, \omega(u\chi_{K^{-1}K} \cdot w) \rangle|. \end{aligned}$$

The first summand is less than $C\varepsilon$ for some constant C, for n large enough , as $\{Nv_n\}$ is bounded in norm. As $\omega(u^{\pi}_{kj} \cdot w) = \omega w^{\pi}_{kj}$ by (4.4) for all k, j, π, we have by (5.4)

$$\langle (Nv_n - 1) \cdot T, \omega(u^{\pi}_{kj}\chi_{K^{-1}K} \cdot w) \rangle = \langle (Nv_n - 1)w^{\pi}_{kj} \cdot T, \omega \rangle \to 0, \; n \to \infty$$

and hence $|\langle (Nv_n - 1) \cdot T, \omega(u\chi_{K^{-1}K} \cdot w) \rangle| < \varepsilon$ for n large enough. Thus,

$$\langle (Nv_n - 1)w \cdot T, \omega \rangle \to 0$$

for each $\omega \in T(G)$. Hence $(Nv_n)w\omega \to w\omega$ weakly for each $\omega \in T(G)$. As the set of all linear combinations of $w\omega$, where $w \in V^{\infty}$, $w = 0$ on E^* and $\omega \in T(G)$, is dense in $\Phi(E)$ [ShT2, Proposition 5.3], $(Nv_n)\omega \to \omega$ weakly for any $\omega \in \Phi(E)$. Taking if necessary convex linear combinations of the Nv_n's, we obtain elements $w_n \in V^{\infty}(G)$, $n \in \mathbb{N}$, such that $\|w_n\omega - \omega\|_{T(G)} \to 0$ as n tends to infinity. Clearly, the sequence $\{w_n\}_n$ satisfies the necessary conditions. $\square$

Corollary 5.5. *Any closed subgroup H of G is a local Ditkin set.*

Proof. We have $H^* = \{(s,t) : f(s) = f(t)\}$, where $f : G \to G\backslash H$, $t \mapsto tH$. As G is metrizable, by Hewitt and Ross [HRI, (8.14)], $G\backslash H$ is metrizable. The statement now follows from Proposition 5.1 and Theorem 5.4. $\square$

For G amenable the statement was obtained in [FKLS] showing that $I_A(H)$ has a bounded approximate identity. For arbitrary locally compact G and H a closed neutral subgroup it was proved in [DD]. Note that there are closed subgroups of l.c.s.c. groups which are not neutral (see [RD, p. 107]).

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References

[A] W. Arveson, Operator algebras and invariant subspaces, Ann. Math. (2) 100 (1974) 433–532.

[BSm] D.P. Blecher, R.R. Smith, The dual of the Haagerup tensor product, J. London Math. Soc. (2) 45 (1) (1992) 126–144.

[Ca] J. de Canniere, Unbounded approximate units of the Fourier algebra of $SO_e(n,1)$: operator algebras and group representations, Proceedings of the International Conference, Neptun/Rom., vol. I, 1980, Monogr. Stud. Math. 17 (1984) 143–146.

[CaHaa] J. de Canniere, U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, Amer. J. Math. 107 (1984) 455–500.

[CoHaa] M. Cowling, U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank 1, Invent. Math. 96 (1989) 507–549.

[DD] J. Delaporte, A. Derighetti, Invariant projections and convolution operators, Proc. Amer. Math. Soc. 129 (5) (2001) 1427–1435.

[Di] J. Dixmier, Les C^*-algèbres et leurs représentations, 2e ed., Cahiers Scientifiques, vol. 29, Gauthier-Villars XV, Paris, 1969, 390p.

[D] S.W. Drury, On non-triangular sets in tensor algebras, Stud. Math. 34 (1970) 253–263.

[EfR] E.G. Effros, Z.J. Ruan, Operator Spaces, London Mathematical Society Monographs, New Series, vol. 23, The Clarendon Press, Oxford University Press, New York, 2000, xvi+363pp.

[ErKSh] J.A. Erdos, A. Katavolos, V.S. Shulman, Rank one subspaces of bimodules over maximal abelian selfadjoint algebras, J. Funct. Anal. 157 (2) (1998) 554–587.

[E] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France 92 (1964) 181–236 (in French).

[FKLS] B. Forrest, E. Kaniuth, A.T. Lau, N. Spronk, Ideals with bounded approximate identities in Fourier algebras, J. Funct. Anal. 203 (1) (2003) 286–304.

[F] J. Froelich, Compact operators, invariant subspaces, and spectral synthesis, J. Funct. Anal. 81 (1) (1988) 1–37.

[Haa] U. Haagerup, Group C^*-algebras without the completely bounded approximation property, unpublished manuscript.

[HaaKr] U. Haagerup, J. Kraus, Approximation properties for group C^*-algebras and group von Neumann algebras, Trans. Amer. Math. Soc. 344 (2) (1994) 667–699.

[He] C. Herz, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (3) (1973) 91–123.

[HRI] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, vol. I, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Band 115, Springer, Berlin, New York, 1979, ix+519pp.

[HRII] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, vol. II, Die Grundlehren der Mathematischen Wissenschaften, Band 152, Springer, New York, Berlin, 1970, ix+771pp.

[RD] W. Roelcke, S. Dierolf, Uniform Structures on Topological Groups and their Quotients, McGraw-Hill International Book Company, New York, 1981 p. 276.

[ShT1] V. Shulman, L. Turowska, Operator synthesis, I: synthetic sets, bilattices and tensor algebras, J. Funct. Anal. 209 (2) (2004) 293–331.

[ShT2] V. Shulman, L. Turowska, Operator synthesis, II: individual operator synthesis and linear operator equations, preprint 2004, Department of Mathematics, Chalmers University of Technology and Göteborg University, Sweden.

[Sm] R.R. Smith, Completely bounded module maps and the Haagerup tensor product, J. Funct. Anal. 102 (1991) 156–175.

[S] N. Spronk, Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras, Proc. London Math. Soc. (3) 89 (1) (2004) 161–192.

[ST] N. Spronk, L. Turowska, Spectral synthesis and operator synthesis for compact groups, J. London Math. Soc. (2) 66 (2) (2002) 361–376.

[T] M. Takesaki, Theory of Operator Algebras I, Springer, Berlin, 1979.

[TT] M. Takesaki, N. Tatsuuma, Duality and subgroups, II, J. Funct. Anal. 11 (1972) 184–190.

[To] I. Todorov, Synthetic properties of ternary masa-bimodules, Houston J. Math., to appear.

[V] N.Th. Varopoulos, Tensor algebras and harmonic analysis, Acta Math. 119 (1967) 51–112.